

Octonions and vacuum stability

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Abstract

The paper addresses one of nontrivial octonion related facts. According to paper gr-qc/0409095, the most stable space-time state is the one described by real Dirac matrices in 11-dimensional space of signature 1 (−) & 10 (+). The internal subspace is 7-dimensional, and its stability is due to a high “zero” energy packing density when using an oblique-angled basis from fundamental vectors of lattice E_8 for the spinor degrees of freedom. The nontrivial fact consists in the following: Dirac symbols with octonion matrix elements can be used to describe states of the space of internal degrees of freedom if and only if the space corresponds either to stable vacuum states or states close to the just mentioned ones. The coincidence of the internal space dimension and signature for absolutely different and independent approaches to the consideration of this issue seems to predetermine the internal space vacuum properties and the apparatus, which is able to constitute the basis of the unified interaction theory.

1. Introduction

4-dimensional Riemannian space with metric tensor $g_{\alpha\beta}$ is considered. The Greek letters take on values $\alpha, \beta, \dots = 0, 1, 2, 3$. The metric tensor is assumed nonsingular, so Dirac symbols (DS) γ_α according to

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2g_{\alpha\beta} \quad (1)$$

can be introduced at every point. This paper considers algebraic properties of DS at some spatial point. The coordinate system is assumed to be locally Cartesian and tensor $g_{\alpha\beta}$ equal to

$$g_{\alpha\beta} = \text{diag}(-1, 1, 1, 1). \quad (2)$$

In the DS theory there are a number of problems, the solution to which directly affects the physical interpretation of DS involving constructions, however, such that there is no complete understanding in regard to their solution method. Mention two of them.

It is well known that symbols $\{\gamma_\alpha\}$ can be realized as Dirac matrices (DM) above any number field (real, complex, quaternion numbers) as well as above the octonion body. The realization in the form of square matrices 4×4 is meant. On the other hand, DS can be realized in the form of real square matrices $N \times N$, $N \geq 4$. Of interest is the question: What is the relation between these two realization types? In particular, what are the characteristics of the subspaces of internal degrees of freedom that are introduced additionally in each complication of the number body used?

For physics, the octonion realization is of a special interest, in particular, for the reason that using any number body except for the octonion one does not allow us even to pose the question of explanation of the irreversibility of actual processes on the basis of time-reversible fundamental laws. The irreversibility phenomenon may be explained only in transition to the formulation of the physical laws in terms of octonions. But in this most interesting case some of the theorems do not hold, on the basis of which the polarization density matrix is introduced and conclusions on the correspondence between tensors and bispinors are reached. The problem is to give an answer to the question: To what extent are those results for the correspondence between tensors and bispinors, which have been found for real matrix realizations of DM, valid for the octonion DM?

This paper makes an attempt to give answers to the two above-formulated problems.

2. DM realization above a real field in Riemannian spaces of a dimension higher than four

The Riemannian spaces of dimension $n \geq 4$ have been studied in connection with construction of matrix spaces (MS), that is the Riemannian spaces, in which the internal degrees of freedom properties are introduced through Dirac matrices γ_A . Subscripts A, B, \dots take on values $A, B = 1, 2, \dots, n$, while the γ_A 's themselves are realized as square matrices $N \times N$ and satisfy relations

$$\gamma_A \gamma_B + \gamma_B \gamma_A = 2g_{AB} \cdot E \quad (3)$$

Here E is the unit matrix in the space of internal degrees of freedom.

The internal degrees of freedom are related, first, with transformations

$$\gamma_A \rightarrow \gamma'_A = S(x) \gamma_A S^{-1}(x), \quad (4)$$

and, second, with the transition to Riemannian spaces of larger dimensions and different signatures.

The MS theory in multidimensional Riemannian spaces with real realizations of DM is discussed in detail in refs. [1]-[3]. These papers also prove the following:

The realization of DM above a real field frequently entails the notion of so-called maximum MS, in which the set of the quantities, the generatrices for which are DM, coincides with the set of all matrices of a given dimension. The maximum MS have odd dimension n .

$$n = 2k + 1,$$

where k is a positive integer. Their signature is therewith of form $(k+1)(+)$ & $k(-)$ or differs from that by a number of “minuses”, which is a multiple of four.

The DM, which can be introduced in the Riemannian space possessing the above properties are square matrices $N \times N$, with N relating to k as

$$N = 2^k.$$

In any MS, either anti-Hermitizing matrix D or Hermitizing matrix C can be introduced. The D or C are determined as

$$D\gamma_a D^{-1} = -\gamma_a^+; \quad C\gamma_a C^{-1} = \gamma_a^+; \quad a = 0, 1, 2, \dots, N-1.$$

The matrix D or matrix C can be used to introduce a Hermitean matrix set, whose existence, in its turn, is needed to introduce the concept of polarization density matrix.

3. Complex numbers and quaternions

The results relating to determination of properties of those multidimensional Riemannian spaces, in which the real DM algebra is mapped isomorphously to the DM algebra in 4-dimensional space of signature $(-+++)$ in realization of the latter above the real, complex and quaternionic fields are summarized in Table 1.

Table 1. Parameters characterizing the isomorphism between DM realized above different number fields and DM realized above the real number field

A method for satisfaction of determining relation $\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2g_{\alpha\beta}$ with $\alpha = 0, 1, 2, 3$ and signature $(-+++)$	
With the help of real matrices	With the help of matrices 4×4 , but using different number bodies
Matrices 4×4 . 16 parameters.	Real number field
Subset of matrices $(4 \times 4) \otimes (2 \times 2)$. 32 parameters. In this realization method, an additional internal subspace of dimension 1 is actually introduced. Gauge group $U(1)$	Complex field. The transition to the matrix notation is performed using isomorphism $I \Leftrightarrow E_{4 \times 4} \otimes i\sigma_2$

Subset of matrices $(4 \times 4) \otimes (4 \times 4)$. 64 parameters. In this realization method, an additional internal subspace of dimension 3 is actually introduced. Gauge group $SU(2)$	Quaternion field. The transition to the matrix notation is performed using isomorphism $\left. \begin{aligned} I_1 &\Leftrightarrow E_{4 \times 4} \otimes i\rho_2\sigma_1 \\ I_2 &\Leftrightarrow E_{4 \times 4} \otimes i\sigma_2 \\ I_3 &\Leftrightarrow E_{4 \times 4} \otimes i\rho_2\sigma_3 \end{aligned} \right\}$
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As it follows from Table 1, the dimension of the internal space that appears in the transition from one number field to another is the same as the number of imaginary units in the number field. The Riemannian spaces are therewith subspaces of maximum MS.

4. Octonion DS

Irrespective of the fact that octonions are discussed extensively in the literature (see, e.g., [4], [5]), nevertheless, here we present some information about these unusual numbers. Naturally, we will do this briefly and only to the extent, which is needed for the consistency of the discussion.

The algebra of octonion imaginary units $\{e_N\}$ is determined as

$$e_M e_N = -\delta_{MN} e_0 + C_{MNK} e_K \quad (5)$$

Here: $M, N, K = 1, 2, \dots, 7$; C_{MNK} are quantities completely antisymmetric in their indices; nonzero components are:

$$C_{123} = C_{145} = C_{246} = C_{347} = C_{176} = C_{257} = C_{365} = 1. \quad (6)$$

Quantity $\Delta[A, B, C]$ is called the associator of three octonions A, B, C :

$$\Delta[A, B, C] = \frac{1}{2} \{(AB)C - A(BC)\}. \quad (7)$$

The whole specificity of the octonion algebra against the matrix algebra is that the associators (7) are nonzero.

Perform the linear real transformation of symbols e_M of the following form:

$$\left. \begin{aligned} e_M &\rightarrow e'_M = G_{MN} \cdot e_N \\ e_0 &\rightarrow e'_0 = e_0 \end{aligned} \right\}. \quad (8)$$

Consider properties of tensor G_{MN} in the 7-dimensional Euclidean space, in which the base vectors are symbols e_M . The substitution of e'_M into

$$e'_M e'_N = -\delta_{MN} e_0 + C_{MNK} e'_K, \quad (9)$$

which symbols e'_M should satisfy, leads to the following two relations:

$$\left. \begin{aligned} G_{MK} G_{NK} &= \delta_{MN} \\ G_{MA} G_{NB} C_{ABC} &= C_{MNS} G_{SC} \end{aligned} \right\}. \quad (10)$$

Quantities G_{MN} produce 14-parametric group G_2 of rank 2. According to the universally adopted classification, group G_2 is attributed to the exceptional Lie group category. Detailed information about the group G_2 can be found, e.g., in ref. [6].

It is known in advance that in the case of DS realization above the octonion body any isomorphous mapping of the appearing DS apparatus to the matrix apparatus cannot exist in principle. So the question is quite appropriate: Do the matrix realizations of DS have any bearing on the octonion DS whatsoever?

To answer this question, make it our aim to construct the DS realization in the form of real DM in a multidimensional Riemannian space, which would satisfy the following requirement:

- When in algebraic operations with octonion DS C_{ABC} play actually no role, the algebraic operations should map to the algebra of real DM of an appropriate dimension. This is true for the algebraic operations with DS $\{\gamma_\alpha\}$ near real DM $\{\bar{\gamma}_\alpha\}$.

To meet this requirement, suppose that in the scheme under discussion there is the smallness parameter $0 < \lambda \ll 1$, such that all matrix elements $(\gamma_\alpha - \bar{\gamma}_\alpha)$ modulo are of the order of λ . Write the matrices γ_α as

$$\gamma_\alpha = \bar{\gamma}_\alpha + f_{\alpha;0} \cdot e_0 + f_{\alpha;N} \cdot e_N; \quad (N = 1, 2, \dots, 7) \quad (11)$$

Matrices (11) will satisfy relation (1), if small matrices $\{f_{\alpha;0}, f_{\alpha;N}\}$ are of the form

$$f_{\alpha;0} = [s_0, \bar{\gamma}_\alpha]_-; \quad f_{\alpha;N} = [s_N, \bar{\gamma}_\alpha]_-, \quad (12)$$

where $\{s_0, s_N\}$ are arbitrary small real matrices 4×4 . Upon substitution of (12) into (11) it turns out that octonion DS are written in the form

$$\gamma_\alpha = \bar{\gamma}_\alpha + [s_0, \bar{\gamma}_\alpha]_- \cdot e_0 + [s_N, \bar{\gamma}_\alpha]_- \cdot e_N. \quad (13)$$

The substitution of (13) into (1) shows that in the first smallness order the C_{ABC} drop out and play no role. This means that the algebra with generatrices satisfying relation

$$[e_M, e_N]_+ = -2\delta_{MN}e_0, \quad (14)$$

$$[e_M, e_N]_- = 2C_{MNK}e_K, \quad (15)$$

can be mapped in the first order of smallness to the algebra of real DM, in which instead of seven imaginary units $\{e_N\}$, seven matrix imaginary units $\{I_N\}$ are used. The specific form of the real DM satisfying either above-formulated requirement can be as follows:

$$\left. \begin{aligned} e_1 &\Leftrightarrow I_1 = E_{4 \times 4} \otimes i\rho_2\sigma_1 \otimes \sigma_1 & e_2 &\Leftrightarrow I_2 = E_{4 \times 4} \otimes i\sigma_2 \otimes \sigma_1 \\ e_3 &\Leftrightarrow I_3 = E_{4 \times 4} \otimes i\rho_2\sigma_3 \otimes \sigma_1 & e_4 &\Leftrightarrow I_4 = E_{4 \times 4} \otimes i\rho_1\sigma_2 \otimes \sigma_3 \\ e_5 &\Leftrightarrow I_5 = E_{4 \times 4} \otimes i\rho_2 \otimes \sigma_3 & e_6 &\Leftrightarrow I_6 = E_{4 \times 4} \otimes i\rho_3\sigma_2 \otimes \sigma_3 \\ e_7 &\Leftrightarrow I_7 = E_{4 \times 4} \otimes E_{4 \times 4} \otimes i\sigma_2. \end{aligned} \right\} \quad (16)$$

The resultant multidimensional Riemannian space has dimension 11 and signature 1 (-) & 10 (+). The DM in the space is written as:

$$\left. \begin{aligned} \bar{\gamma}_0 &= -i\rho_2\sigma_1 \otimes E_{4 \times 4} \otimes E_{2 \times 2}; & \bar{\gamma}_1 &= \rho_1 \otimes E_{4 \times 4} \otimes E_{2 \times 2} \\ \bar{\gamma}_2 &= \rho_2\sigma_2 \otimes E_{4 \times 4} \otimes E_{2 \times 2}; & \bar{\gamma}_3 &= \rho_3 \otimes E_{4 \times 4} \otimes E_{2 \times 2}; \\ \bar{\gamma}_4 &= i\rho_2\sigma_3 \otimes i\rho_2\sigma_1 \otimes \sigma_1; & \bar{\gamma}_5 &= i\rho_2\sigma_3 \otimes i\sigma_2 \otimes \sigma_1 \\ \bar{\gamma}_6 &= i\rho_2\sigma_3 \otimes i\rho_2\sigma_3 \otimes \sigma_1 & \bar{\gamma}_7 &= i\rho_2\sigma_3 \otimes i\rho_1\sigma_2 \otimes \sigma_3 \\ \bar{\gamma}_8 &= i\rho_2\sigma_3 \otimes i\rho_2 \otimes \sigma_3; & \bar{\gamma}_9 &= i\rho_2\sigma_3 \otimes i\rho_3\sigma_2 \otimes \sigma_3; \\ \bar{\gamma}_{10} &= i\rho_2\sigma_3 \otimes E_{4 \times 4} \otimes i\sigma_2. \end{aligned} \right\} \quad (17)$$

Expressions (13) have the meaning of the ones in the first order of smallness for the first four DM among eleven DM. The expressions for all the eleven DM in the first order of smallness are derived from

$$\gamma_\alpha = \bar{\gamma}_\alpha + [s, \bar{\gamma}_\alpha]_-; \quad \gamma_{N+3} = \bar{\gamma}_{N+3} + [s, \bar{\gamma}_{N+3}]_-; \quad N = 1, 2, \dots, 7, \quad (18)$$

where

$$s = s_0 \cdot E + s_N \cdot I_N. \quad (19)$$

Thus, in the linear approximation the octonion DM γ_α can be treated as ordinary matrices, if for the basic matrices, in the vicinity of which the expansion proceeds, real DM are used in 11-dimensional Riemannian space of signature 1(-) & 10(+). Pay attention to the fact that except for the reality no other properties of DM in 11-dimensional Riemannian space have been used in this consideration. This means that instead of system (17) that DM system can be used in the consideration, which has been derived in [7] from system (17) through transition to the oblique-angled basis produced by simple root vectors of Lie algebra E_8 .

In the general case the following rule remains valid: If it was possible to realize DS with the help of octonion DM 4×4 , then after that one can transfer from one realization to another using transformations G_2 .

5. Discussion

Although the octonion DS can be written in the form of matrices 4×4 in the general case, but the algebra of the matrices possesses no associativity and, hence, cannot be mapped to the algebra of ordinary real matrices in a multidimensional space. In the linear approximation, however, the algebra of octonion DM is mapped to that of real DM in 11-dimensional Riemannian space of signature 1(-) & 10(+). One of possible DM systems in this space is of form (17).

The result obtained is of interest for several reasons.

Reason 1 is that the correspondence found by us between octonion DS and real DM in a multidimensional Riemannian space leads to the Riemannian space, in which the most stable vacuum state appears. Ref. [7] shows that the most stable vacuum state both among the internal subspaces of dimensions other than 7 and among DM of different spinor basis structure is the DM realization in the form of real matrices, in which the oblique-angled basis from the set of fundamental vectors of lattice E_8 is used. In this realization, the

internal space dimension is 7; the specific form of the lattice DM is presented in ref. [7] and the matrix of transition from the orthonormal basis to the lattice one is given, e.g., in [7], [8].

Reason 2 is that using any number body, except for the octonion body, in physical theories does not allow us even to pose the question of explanation of the irreversibility of actual processes on the basis of time-reversible fundamental laws with writing the latter in terms of any number field. The irreversibility phenomenon may be explained only in transition to the formulation of the physical laws in terms of octonions.

In this connection note that it has been long since the physicists have paid attention to the existence of an evident contradiction: on the one hand, the dynamic equations describing fundamental interactions possess time reversibility; on the other hand, actual processes that occur in the Nature are irreversible. R. Penrose in [9] writes: "...It is hard to understand how our immense Universe could "sink" into one or another of the states with being unable to even imagine in what time direction to start! ...the only explanation ... remains: not all accurate physics laws are symmetric in time!...".

If DM are realized above the octonion body, then the transition amplitudes automatically cease to be associative. While this just means that the reversibility in time does disappear at the level of fundamental processes in the microworld. In fact, if A_1, A_2, A_3, \dots are amplitudes of the transitions from initial state t_0 to states arising at times t_1, t_2, t_3, \dots , then the amplitude for one of the paths of the process proceeding in the time-forward direction should be found according to rule

$$((A_1 \cdot A_2) \cdot A_3) \dots, \quad (20)$$

while the conjugate amplitude for the process running in the time-backward direction should be found according to rule

$$(A_1 \cdot (A_2 \cdot A_3 \dots)). \quad (21)$$

At the level of real, complex and quaternionic numbers expressions (20), (21) lead to the same probabilities of transitions. But as soon as octonions come into use, the equality between expressions (20), (21), generally speaking, disappears. Moreover, the body of octonion numbers is the only one possessing this property. This means that we may necessarily resort to the octonion quantities for explanation of the irreversibility of processes.

The above considerations and results justify the multiple attempts to consider the octonion wave functions for half-integer spin particles. We only point out to refs. [5], [10], [11] as typical papers from the standpoint of the method for consideration of octonion Dirac matrices. The method of these papers is valid only to the quadratic approximation, as in these papers there is either explicit or implicit transition to so-called split octonions (introduction of the outer imaginary unit commuting with all octonions) or the octonion composition rule is replaced by the open product. Similar (or equivalent) techniques restore the associativity of the modified number body and allow the standard matrix apparatus to be employed. However, in so doing a most interesting part of the octonion specificity is lost.

A method for description of the half-integer spin particle dynamics is the method of mapping of tensors to bispinors developed in a number of papers (see, e.g., [3]). In the method, one of principal objects is bispinor matrix Z . For the octonion implementation of DS, the matrix Z exists in the linear approximation and, as it follows from (13), coincides with S^{-1} . Through multiplication on the right by the projectors, states with different quantum numbers can be separated from the bispinor matrix. For example, one of the subgroups of group G_2 is $SU(3)$. In the general case there is no bispinor matrix, however, the results obtained using the methods for consideration of the transformations of DM 4×4 , which are suggested in ref. [12], remain valid.

Thus, the vacuum stability requirement can be made consistent with using the most general number body. In so doing any violation of the bounds of the 7-dimensional internal Euclidean space will result in vacuum instability (and appearance of tachyons as a consequence).

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